On Harmonic Curvatures of Null Curves of the AW(k)-Type in Lorentzian Space

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We investigate null curves of the AW(k)-type ($1 \leq k \leq 3$) in the 3-dimensional Lorentzian space, L^3, and give curvature conditions of these curves by using the Cartan frame. Moreover, we study harmonic curvatures of curves of AW(k)-type and show that if the α Frenet curve is of type AW(1), then α is a null helix.

Key words: Null Curve; Harmonic Curvature; Frenet Formulas; AW(k)-Type Curve.
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1. Introduction

The theory of space curves of a M Riemannian manifold is fully developed and its local and global geometry are well-known. In case of M is proper semi-Riemannian, there are three categories of curves, namely spacelike, timelike and null. The study of timelike curves has many similarities with that of spacelike curves. However, null curves have many properties very different from spacelike or timelike curves. In other words, null curve theory has many results which have no Riemannian analogues. Seeing that, the induced metric of a null curve is degenerate; this case is much more complicated and also different from the non-degenerate case. Motivated by this fundamental observation, the book by Duggal and Benjancu [1] which is called "a lightlike submanifolds of Semi-Riemannian manifolds and their applications to relativity" is very important for null curves and surfaces (in particular, null congruences) in mathematical physics.

Null hypersurfaces play an important role in the study of various problems of both electromagnetism and relativity as well as of mathematics and physics of gravitation [2]. On the other hand their geometry is completely different from the classical geometry of non-degenerate submanifolds. A starting point to study null surfaces, or in general null hypersurfaces, consists of investigating the curves that live in those hypersurfaces. In this sense, the null curves in Lorentzian space forms have been studied by both mathematicians and physicists [3 – 5].

The fact that relativity theory is expressed in terms of Lorentz geometry is lucky for geometers, who can thus penetrate surprisingly quickly into cosmology (redshift, expanding universe, and big bang) and a topic no less interesting geometrically, the gravitation of a single star (perihelion procession, bending of light, and black holes).

Furthermore, in relativity theory, a lightlike particle is a future-pointing null geodesic [2]. In addition, we can see other examples in [6] and [7] which show a particle model entirely based on geometry of null curves in physics.

From the differential geometric point of view, the study of null curves has its own interest. Many interesting results on null curves have been obtained by many mathematicians (see [8 – 12]). But the geometry of null curves is different. When we study these curves, some difficulties arise because the arc length vanishes, so that it is not possible to normalize the tangent vector in the usual way. Thus, a new parameter called the pseudo-arc which normalizes the derivative of the tangent vector is introduced. Many authors defined a Frenet frame with the minimum number of curvature functions (called the Cartan frame) for a null curve in an n-dimensional Lorentzian space form.

In this paper, another important subject are AW(k)-type curves. Many studies on curves of AW(k)-type have been done by many mathematicians. For example, in [13] and [14], the authors gave curvature conditions and characterizations related to AW(k)-type curves in

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E^n, and in [15] the authors investigated curves of AW(k)-type in the 3-dimensional null cone.

A literature survey indicated that there are no null curves concerning curves of AW(k)-type. The main purpose of this paper is to propose such an approach on the basis of null curves of AW(k)-type in L^3.

In this paper, null curves of AW(k)-type are studied in the 3-dimensional Lorentzian space, L^3. In order to give curvature conditions of AW(k)-type curves, the Duggal's Frenet equations are introduced in the Cartan frame with respect to the distinguished parameter p [1]. Furthermore, by considering first the harmonic curvature of a null generalized helix as given in [9], some theorems are given, which are related to harmonic curvatures of Frenet curves of AW(k)-type.

2. Preliminaries

The Lorentzian space L^3 is defined as the vector space IR^3 endowed with the Lorentzian metric

$$\langle\cdot,\cdot\rangle : -\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}x_3^2,$$

where (x_1,x_2,x_3) are canonical coordinates in IR^3. A tangent vector v of L^3 is said to be

- spacelike, if $\langle v,v\rangle > 0$ or $v=0$;
- timelike, if $\langle v,v\rangle < 0$;
- lightlike or null, if $\langle v,v\rangle = 0$ and $v\neq 0$.

A null frame of L^3 is a positively oriented ordered triple (λ,N,W) of vectors satisfying

$$\langle\lambda,\lambda\rangle=\langle N,N\rangle=0,\quad \langle\lambda,N\rangle=1,$$
$$\langle\lambda,W\rangle=\langle N,W\rangle=0,\quad \langle W,W\rangle=1.$$

Let α be a null curve in L^3, i.e. $\left\langle\frac{\mathrm{d}\alpha}{\mathrm{d}s},\frac{\mathrm{d}\alpha}{\mathrm{d}s}\right\rangle=0$ and $\frac{\mathrm{d}\alpha}{\mathrm{d}s}\neq 0$. Now suppose that α is framed by $F=(\lambda,N,W)$ with $\lambda=\frac{\mathrm{d}\alpha}{\mathrm{d}s}$. Then the vector fields N and W define line bundles $ntr(\alpha)$ and $S(\alpha^{\perp})$ over α, respectively. The line bundle $S(T\alpha^{\perp})$ is called the screen vector bundle and $ntr(\alpha)$ the null transversal vector bundle of α with respect to $S(\alpha^{\perp})$, respectively.

The curve α is called a Frenet curve of oscillating order 3, if its derivatives $\alpha'(s)$, $\alpha''(s)$; $\alpha'''(s)$ are linearly independent and $\alpha'(s)$, $\alpha''(s)$, $\alpha'''(s)$, $\alpha''''(s)$ are no longer linearly independent for all $s\in I$. To each Frenet curve of order 3 one can associate an orthonormal 3-frame λ,N,W along α [such that $\alpha'(s)=\lambda$] called the Frenet frame, such that the Frenet formulas are defined as follows [1]:

$$\frac{\mathrm{d}\lambda}{\mathrm{d}s}=h\lambda+\kappa_1 W,$$
$$\frac{\mathrm{d}N}{\mathrm{d}s}=-hN+\kappa_2 W,\tag{1}$$
$$\frac{\mathrm{d}W}{\mathrm{d}s}=-\kappa_2\lambda-\kappa_1 N.$$

The functions h, κ_1, κ_2 are called the curvature functions of α.

There always exists a parameter p of α such that $h=0$ in (1). This parameter p is called a distinguished parameter of α which is uniquely determined for prescribed screen vector bundle up to affine transformation [1]. Hence, it is possible to write

$$\ell(p)=\frac{\mathrm{d}\alpha}{\mathrm{d}p}(p),\;\; n(p)=-N(p),\;\; u(p)=W(p),$$

and the Frenet formula of α with respect to $F=(\ell,n,u)$ become

$$\ell'=\kappa_1 u,$$
$$n'=-\kappa_2 u,\tag{2}$$
$$u'=-\kappa_2\ell+\kappa_1 n.$$

Here the prime "$'$" denotes differentiation with respect to p. The null frame F is called the Cartan frame of $\alpha(p)$. A parametrized null curve parametrized by the distinguished parameter p together with its Cartan frame is called a Cartan framed null curve.

If $\det(\lambda,N,W)>0$, then Cartan frames are negatively oriented, that is, $\det(\ell,n,u)<0$.

For a general theory of parametrized null curves, the reader is referred to [1].

Definition 2.1. A null curve with respect to a Cartan frame with $\kappa_2=0$ is called a generalized null cubic [1].

Proposition 2.2. Let α be a Frenet curve of L^3 of osculating order 3, then we have

$$\alpha'(p)=\ell(p),$$
$$\alpha''(p)=\kappa_1 u,\tag{3}$$

$$\alpha'''(p)=-\kappa_1\kappa_2\ell+\kappa_1\kappa_2 n+\kappa_1' u,\tag{4}$$

$$\begin{aligned}\alpha''''(p)=&(-\kappa_1\kappa_2'-2\kappa_1'\kappa_2)\ell\\&+(-\kappa_1'\kappa_1+\kappa_1'\kappa_2+\kappa_1\kappa_2')n\\&+(\kappa_1''+\kappa_1^2\kappa_2-\kappa_1\kappa_2^2)u.\end{aligned}\tag{5}$$

Notation 2.3. Let us write

$$N_1(p) = \kappa_1 u, \tag{6}$$

$$N_2(p) = \kappa_1\kappa_2 n + \kappa_1' u, \tag{7}$$

$$\begin{aligned} N_3(p) = &(-\kappa_1'\kappa_1 + \kappa_1'\kappa_2 + \kappa_1\kappa_2')n \\ &+ (\kappa_1'' + \kappa_1^2\kappa_2 - \kappa_1\kappa_2^2)u. \end{aligned} \tag{8}$$

Corollary 2.4. $\alpha'(p)$, $\alpha''(p)$, $\alpha'''(p)$, $\alpha''''(p)$ are linearly dependent, if and only if $N_1(p)$, $N_2(p)$, $N_3(p)$ are linearly dependent.

3. Curves of AW(k)-Type

In this part we consider Frenet curves of AW(k)-type.

Definition 3.1. Frenet curves are

(i) of type AW(1), if they satisfy $N_3(p) = 0$;

(ii) of type AW(2), if they satisfy

$$\|N_2(p)\|^2 N_3(p) = \langle N_3(p), N_2(p)\rangle N_2(p); \tag{9}$$

(iii) of type AW(3), if they satisfy

$$\|N_1(p)\|^2 N_3(p) = \langle N_3(p), N_1(p)\rangle N_1(p). \tag{10}$$

Proposition 3.2. Let α be a Frenet curve of order 3. Then α is of type AW(1), if and only if

$$-\kappa_1'\kappa_1 + \kappa_1'\kappa_1 + \kappa_1\kappa_2' = 0 \tag{11}$$

and

$$\kappa_1'' + \kappa_1^2\kappa_2 - \kappa_1\kappa_2^2 = 0. \tag{12}$$

Proof. Let α be a curve of type AW(1). From definition 3.1.(i) $N_3(p) = 0$. Then from equality (8) we have

$$\begin{aligned} &\left\{-\kappa_1'\kappa_1 + \kappa_1'\kappa_2 + \kappa_1\kappa_2'\right\} n \\ &+ \left\{\kappa_1'' + \kappa_1^2\kappa_2 - \kappa_1\kappa_2^2\right\} u = 0. \end{aligned}$$

Furthermore, since n and u are linearly independent, one can obtain (11) and (12). Since the converse statement is trivial, the proof is completed.

A generalized null cubic can be given as an example for proposition 3.2.

Proposition 3.3. Let α be a Frenet curve of order 3. Then α is of type AW(2), if and only if

$$\begin{aligned} &-(\kappa_1')^3\kappa_1 + (\kappa_1')^3\kappa_2 + \kappa_1(\kappa_1')^2\kappa_2' = \\ &\kappa_1\kappa_2\kappa_1'\kappa_1'' + \kappa_1^3\kappa_1'\kappa_2^2 - \kappa_1^2\kappa_1'\kappa_2^3. \end{aligned} \tag{13}$$

Proof. If α is of type AW(2), (9) holds on α. Substituting (7) and (8) into (9), one can obtain (13). The converse statement is trivial. This completes the proof.

Example 3.4. Let us consider an example [1, 16] as the curve α in R_1^3 given by

$$x_0 = \sinh s,\ x_1 = s,\ x_2 = \cosh s,\ s \in \mathbb{R}.$$

Then we choose the Frenet frame $F = \{\lambda, N, W\}$ as follows:

$$\begin{aligned} \lambda &= (\cosh s, 1, \sinh s), \\ N &= \frac{1}{2}(-\cosh s, 1, -\sinh s), \\ W &= (\sinh s, 0, \cosh s). \end{aligned}$$

Thus from (2), we get $\kappa_1 = 1$ and $\kappa_2 = -\dfrac{1}{2}$. κ_1 and κ_2 hold on (13).

Proposition 3.5. Let α be a Frenet curve of order 3. Then α is of type AW(3), if and only if

$$-\kappa_1'\kappa_1^3 + \kappa_1'\kappa_1^2\kappa_2 + \kappa_1^3\kappa_2' = 0. \tag{14}$$

Proof. Since α is of type AW(3), (10) holds on α. So substituting (6) and (8) into (10), we have (14). The converse statement is trivial. Hence our proposition is proved.

Example 3.6. We consider as an example [1, 16] the curve α in R_1^3 given by the equations

$$\begin{aligned} x_0 &= s,\ x_1 = \frac{1}{b}\sin(bs + a) + c, \\ x_2 &= -\frac{1}{b}\cos(bs + a) + d, \end{aligned}$$

where $a, b \neq 0, c, d$ are real constants. Then we have

$$\begin{aligned} \lambda &= (1, \cos(bs + a), \sin(bs + a)), \\ N &= (0, -\sin(bs + a), \cos(bs + a)), \\ W &= \frac{1}{2}(-1, \cos(bs + a), \sin(bs + a)). \end{aligned}$$

Using (2), we get $\kappa_2 = \frac{b}{2}$ and $\kappa_1 = b$: κ_1 and κ_2 hold on (14).

4. Harmonic Curvature of a Frenet Curve

In this part we consider harmonic curvatures of Frenet curves of AW(k)-type.

Definition 4.1. α is a null helix $\Leftrightarrow H_1 =$ constant.

Definition 4.2. Assume that $\alpha \subset L^3$ is a null generalized helix given by the curvature functions κ_1 and κ_2. Then the first harmonic curvatures of α in L^3 can be written as follows (as in [9]):

$$H_1 = \frac{\kappa_2}{\kappa_1}. \tag{15}$$

Proposition 4.3.

$$\ell' = \kappa_1 u, \tag{16}$$

$$n' = -\kappa_1 H_1 u, \tag{17}$$

$$u' = -\kappa_1 H_1 \ell + \kappa_1 n. \tag{18}$$

Proposition 4.4. Let α be a Frenet curve of L^3 of osculating order 3, then we have

$$\alpha''(p) = \kappa_1 u, \tag{19}$$

$$\alpha'''(p) = -\kappa_1^2 H_1 \ell + \kappa_1^2 H_1 n + \kappa_1' u, \tag{20}$$

$$\begin{aligned}\alpha''''(p) = {} & (-\kappa_1^2 H_1' - 3\kappa_1\kappa_1' H_1)\ell \\ & + (-\kappa_1'\kappa_1 + 2\kappa_1\kappa_1' H_1 + \kappa_1^2 H_1')n \\ & + (\kappa_1'' + \kappa_1^3 H_1 - \kappa_1^3 H_1^2)u.\end{aligned} \tag{21}$$

Notation 4.5. Let us write

$$N_1(p) = \kappa_1 u, \tag{22}$$

$$N_2(p) = \kappa_1^2 H_1 n + \kappa_1' u, \tag{23}$$

$$\begin{aligned}N_3(p) = {} & (-\kappa_1'\kappa_1 + 2\kappa_1\kappa_1' H_1 + \kappa_1^2 H_1')n \\ & + (\kappa_1'' + \kappa_1^3 H_1 - \kappa_1^3 H_1^2)u.\end{aligned} \tag{24}$$

Theorem 4.6. Let α be a Frenet curve of order 3. If α is of type AW(1), then α is a null helix.

Proof. Since α is of type AW(1), from definition 3.1.(i) $N_3(p) = 0$. Thus from (24), we have

$$\begin{aligned}& \{-\kappa_1'\kappa_1 + 2\kappa_1\kappa_1' H_1 + \kappa_1^2 H_1'\}n \\ & + \{\kappa_1'' + \kappa_1^3 H_1 - \kappa_1^3 H_1^2\}u = 0.\end{aligned}$$

Since n and u are linearly dependent, the coefficients of n and u vectors should be zero. Therefore, we get

$$-\kappa_1'\kappa_1 + 2\kappa_1\kappa_1' H_1 + \kappa_1^2 H_1' = 0$$

and

$$\kappa_1'' + \kappa_1^3 H_1 - \kappa_1^3 H_1^2 = 0.$$

If these differential equations are solved, we find $\kappa_1 = \kappa_2 =$ constant, that is, $H_1 =$ constant. Consequently, α is a null helix. This completes the proof.

A generalized null cubic can be given as an example for theorem 4.6.

Theorem 4.7. Let α be a Frenet curve of order 3. Then, if α is of AW(2)-type, we have

$$\begin{aligned}& \kappa_1^2\kappa_1'' H_1 + \kappa_1^5 H_1^2 - \kappa_1^5 H_1^3 + \kappa_1(\kappa_1')^2 \\ & - 2\kappa_1(\kappa_1')^2 H_1 - \kappa_1'\kappa_1^2 H_1' = 0.\end{aligned} \tag{25}$$

Proof. Since α is of AW(2)-type, from equality (9), $N_2(p)$ and $N_3(p)$ are linearly dependent. In (23), we assume that coefficients of n and u are $\gamma(p)$ and $\beta(p)$, respectively, in (24), we assume that coefficients of n and u are $\tau(p)$ and $\delta(p)$, respectively. Thus we can write equalities (23) and (24) as follows:

$$\begin{aligned}N_2(p) &= \gamma(p)n(p) + \beta(p)u(p), \\ N_3(p) &= \tau(p)n(p) + \delta(p)u(p).\end{aligned}$$

Since $N_2(p)$ and $N_3(p)$ are linearly dependent, the coefficients determinant is zero. Thus, we can write

$$\begin{vmatrix}\gamma(p) & \beta(p) \\ \tau(p) & \delta(p)\end{vmatrix} = 0. \tag{26}$$

Here

$$\gamma(p) = \kappa_1^2 H_1, \quad \beta(p) = \kappa_1'$$

and

$$\begin{aligned}\tau(p) &= -\kappa_1'\kappa_1 + 2\kappa_1\kappa_1' H_1 + \kappa_1^2 H_1', \\ \delta(p) &= \kappa_1'' + \kappa_1^3 H_1 - \kappa_1^3 H_1^2.\end{aligned}$$

If we write equalities of $\gamma(p)$, $\beta(p)$, $\tau(p)$, $\delta(p)$ in (26), we have (25).

A generalized null cubic can be given as an example for theorem 4.7.

Theorem 4.8. Let α be a Frenet curve of order 3. α is of type AW(3), if and only if

$$-\kappa_1'\kappa_1 + 2\kappa_1\kappa_1' H_1 + \kappa_1^2 H_1' = 0. \tag{27}$$

Proof. Since α is of AW(3)-type, (10) holds on α. Thus, $N_1(p)$ and $N_3(p)$ are linearly dependent, and we have (27). Conversely, if (27) holds, it is easy to show that α is of AW(3)-type. This completes the proof of the theorem.

Example 4.9. Consider a null curve α of R_1^3 given by the equations

$$x_1 = \frac{1}{2}s\sqrt{s^2+1} + \frac{1}{2}\ln\left|\sqrt{s^2+1}+s\right|,$$
$$x_2 = \frac{1}{2}s^2,\ x_3 = s,\ s \in \mathbb{R}.$$

Then we choose the Frenet frame $F = \{\lambda, N, W\}$ as follows:

$$\lambda = \left(\sqrt{s^2+1}, s, 1\right),$$
$$N = \left(\frac{s}{\sqrt{s^2+1}}, 1, 0\right),$$
$$W = \left(-\frac{1}{2}\sqrt{s^2+1}, -\frac{1}{2}s, \frac{1}{2}\right).$$

Thus from (27), we have $\kappa_1 = 1$ and $\kappa_2 = -\frac{1}{2}$. κ_1 and κ_2 holds on (27).

5. Conclusions

It is well-known that null curves are important in the development of general relativity theory and gravitation in mathematical physics. In this study, null curves of AW(k)-type are examined in the 3-dimensional Lorentzian space, and the curvature conditions of null curves of AW(k)-type are identified by considering the Cartan frame. Furthermore, it is shown that, if α Frenet curve is of type AW(1), then the α is a null helix.

It is hoped that this study about null curves of AW(k)-type in the 3-dimensional Lorentzian space serves researchers who carry out research especially in general relativity and gravitation.

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[1] K. L. Duggal and A. Bejancu, Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Kluwer Academic Publishers, Dordrecht 1996.
[2] B. O'neill, Semi-Riemannian Geometry with Application to Relativity, Academic Press, New York 1983.
[3] R. Capovilla, J. Guven, and E. Rojas, Gen. Relativ. Gravit. **38**, 689 (2006).
[4] A. Ferrandez, A. Gimenez, and P. Lucas, J. Geom. Phys. **45**, 116 (2003).
[5] J. Samuel and R. Nityananda, J. Phys. A: Math. Gen. **33**, 2895 (2000).
[6] A. Ferrandez, A. Gimenez, and P. Lucas, Phys. Lett. B **543**, 311 (2002).
[7] A. Nersessian and E. Ramos, Phys. Lett. B **445**, 123 (1998).
[8] H. Balgetir, M. Bektaş, and J. Inoguchi, Note Mathematica **23**, 7 (2004).
[9] A. F. Yalınız and H. H. Hacısalihoğlu, Math. Comp. Appl. **10**, 105 (2005).
[10] A. C. Çöken and Ü. Çiftçi, Geometriae Dedicata **114**, 71 (2005).
[11] K. L. Duggal and D. H. Jın, Math. J. Toyama Univ. **22**, 95 (1999).
[12] K. L. Duggal, Acta Appl. Math. **95**, 135 (2007).
[13] K. Arslan and C. Özgür, Curves and Surfaces of AW(k)-Type, in: Geometry and Topology of Submanifolds IX (Eds. F. Defever, J. M. Morvan, I. Van De Woestijne, L. Verstraelen, G. Zafindratafa), World Scientific Publishing, Singapore 1999, p. 21.
[14] C. Özgür and F. Gezgin, Differ. Geom. Dyn. Syst. **7**, 74 (2005).
[15] M. Külahcı, M. Bektaş, and M. Ergüt, Phys. Lett. A **371**, 275 (2007).
[16] B. Şahin, E. Kılıç, and R. Güneş, Diff. Geom. Dyn. Syst. **3**, 31 (2001).